

Evolution of entanglement for spin-flip dynamics

Slawomir Koziel

Institute of Experimental Physics, Gdansk University,
Wita Stwosza 57, 80-952 Gdansk, Poland

e-mail: koziel@iftia6.univ.gda.pl

Wladyslaw A. Majewski

Institute of Theoretical Physics and Astrophysics, Gdansk University,
Wita Stwosza 57, 80-952 Gdansk, Poland

e-mail: fizwam@univ.gda.pl

Abstract

A model of evolution of bipartite quantum state entanglement is studied. It involves the recently introduced quantum block spin-flip dynamics on a lattice. We find that for initially separable states the considered evolution leads, in general, to entangled states. We also present a complete characterization of two-point correlation functions for that type of dynamics to confirm enhancement of quantum correlation for the considered system.

1 Introduction

Entanglement¹ first noted by Schrödinger and von Neumann as the characteristic trait of quantum mechanics [19],[20] and by Einstein, Podolsky, Rosen [6] as a quantum superposition of two distinct states of physical systems with the great impact on our understanding of the notion of reality in the atomic scale, is undoubtedly an essential feature of quantum mechanics. That concept can be relatively easily treated for the hamiltonian type evolution. Namely, writing the full Hamiltonian of a composite system, one should specify the interaction part explicitly. We want to emphasize that it is exactly that part of hamiltonian, which is responsible for the evolution of entanglement in the following sense: it is impossible that all factorizable states remain factorizable during the interaction unless the interaction part of the hamiltonian is trivial.

The question of the evolution of entanglement for quantum stochastic semigroups is much harder task. To explain that point in detail, we recall that in recent papers, B. Zegarliński and one of us, proposed a general scheme for a quantization of stochastic dynamics which describe interacting classical particles [10]-[13]. In that scheme, guided by classical theory (cf. [9]), a general recipe for quantum stochastic dynamics of jump type and diffusive type was given (see also [1],[4],[5],[7]). In particular, to define the infinitesimal generators of Markov semigroups, the theory of generalized conditional expectations (in the Accardi-Cecchini sense) in the framework of quantum L_p -spaces was used. In that way, a general scheme to produce, describe and analyze dynamical systems with evolution originating from quantization of stochastic processes and such that their equilibrium states are prescribed (quantum) Gibbs states, was established.

Having such a general plan of quantization of stochastic dynamics one may pose a natural question about its nontriviality. Under this notion we understand, first of all, that infinitesimal generator of such dynamics is not a function of the hamiltonian which can be extracted from the given Gibbs state. This requirement arises in a natural way from the methodology of construction of quantum stochastic dynamics sketched in the preceding paragraph. In fact, it has been shown [14] that generators defined within the presented L_p -space setting satisfy the above specification. On the other hand, the genuine quantum map should produce quantum correlations besides being well defined in non-commutative structures. The important point to note here is the idea of quantum correlations. Recently, the concept of coefficient of quantum correlations was introduced [16] and it was shown that such coefficient is not equal to zero only for non-separable states. In other words the entanglement is closely related to non-classical correlations between two subsystems of a composite system and such correlations arise from nontrivial interactions between the subsystems. Therefore, to have nontrivial quantum dynamics one should show that the dynamical maps under considerations are able to increase entanglement.

The main difficulty in showing an increment of quantum correlations stems from the fact that the explicit form of interactions responsible for the transition rates used in the definition of quantum Markov generator is not known. Consequently, the present case is much more difficult from the hamiltonian one. To start a general analysis of evolution of entanglement for quantum stochastic dynamics, we begin with the block-spin flip type dynamics,

¹We refer the reader not familiar with the notion of entanglement to Appendix A, where some basic facts and definitions concerning separable and entangled states has been collected

which is the main object of interest in this paper. Let us add that this dynamics can serve as a paradigm for a quantization of Glauber dynamics. Thus, we will be concerned with evolution of entanglement in concrete finite dimensional model of quantum block-spin flip dynamics.

In order to carry out the analysis of entanglement we use two different approaches. Having fixed notations and given preliminaries (Section II), in Section III we consider an explicit example of low-dimensional jump-type system, showing that the block spin flip dynamics leads to the entanglement of the initial separable quantum state. In Section IV we study some properly chosen correlation functions and show that an enhancement of quantum correlation is *typical* for the considered dynamics (see Proposition 4.5). Finally, in Section V we will comment the obtained results.

2 Quantum block spin-flip dynamics

In the general approach to quantum jump-type dynamics of quantum systems on a lattice it is convenient to consider firstly the finite volume case - a system associated with a finite region Λ , and then to perform the thermodynamic limit with Λ going to \mathbf{Z}^d , where \mathbf{Z}^d denotes d -dimensional lattice (for mathematical details of algebraic description of quantum statistical mechanics see [3]). However, as we have mentioned, the main scope of the paper is the analysis of evolution of entanglement and increment of entanglement is taken as a signature of nontrivial interactions between two subsystems. This phenomenon should be present both for finite and infinite subsystems. To simplify our exposition as much as possible we restrict to the essential ingredients of the finite volume case (for a general description see [10],[12],[13]). Thus, we shall consider a composite system $I+II$ associated with a region $\Lambda = \Lambda_I \cup \Lambda_{II}$, where Λ_i , $i = I, II$ are disjoint finite subregions of the lattice \mathbf{Z}^d . To have a concrete dynamical system we will describe the construction of the block spin-flip dynamics related to the region Λ . To this end, we associate with Λ_I (Λ_{II}) the finite dimensional Hilbert space $\mathcal{H}_{\Lambda_I} \equiv \mathcal{H}_1$ ($\mathcal{H}_{\Lambda_{II}} \equiv \mathcal{H}_2$) as the space of its pure states, the set of density matrices S_1 (S_2) as the space of all mixed states, and the set of all bounded linear operators $\mathcal{B}(\mathcal{H}_1)$ ($\mathcal{B}(\mathcal{H}_2)$) as the algebras of observables. Thus, the composite system Λ is described by $\mathcal{H}_1 \otimes \mathcal{H}_2$, $S_1 \otimes S_2$ and $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \cong \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, respectively.

The reference state, playing the crucial role for classical and quantum case, is given here by the Gibbs state, i.e. with each region Λ (Λ_I, Λ_{II} , respectively) we associate the Hamiltonian H_Λ ($H_{\Lambda_I}, H_{\Lambda_{II}}$, respectively) and subsequently the corresponding Gibbs state $\omega_\Lambda(\cdot) = Tr \rho_\Lambda(\cdot)$ with $\rho_\Lambda = \frac{e^{-\beta H_\Lambda}}{Tre^{-\beta H_\Lambda}} \equiv \rho$, etc.

Guided by the classical theory (cf. [9]), where conditional expectations serve for the construction of jump type stochastic processes, we use their non-commutative generalizations (in the Accardi-Cecchini sense) to define the infinitesimal generator $\mathcal{L}_{\Lambda, \Lambda_I} \equiv \mathcal{L}$ of the corresponding quantum spin-flip semigroup where the "block spin flip" is carried out on $\Lambda_I (\subseteq \Lambda)$. Therefore let us introduce a map $E_{\Lambda, \Lambda_I} (\equiv E) : \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ defined as follows:

$$E(A) = Tr_1(\gamma^* A \gamma) \quad (1)$$

where

$$\gamma = \rho^{\frac{1}{2}} (Tr_1 \rho)^{-\frac{1}{2}}$$

with Tr_1 denoting the partial trace (over the Hilbert space \mathcal{H}_1). Let us remark, that for infinite region case, γ is defined as analytic extension of non-commutative Radon-Nikodym cocycles (cf. [12]). Using the above defined generalized conditional expectation E we can introduce the following operator \mathcal{L} defined on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by

$$\mathcal{L}(A) = E(A) - A$$

for $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. *The important point to note here is the form of the infinitesimal Markov generator \mathcal{L} : there is no explicit term describing the interactions between regions Λ_I and Λ_{II} and this is the origin of difficulties in the presented analysis.* Given a state ω_ρ , defined by a density matrix ρ , $\omega_\rho = Tr(\rho(\cdot))$, one can define on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ the following scalar product

$$\langle\langle A, B \rangle\rangle_{\omega_\rho} \equiv Tr(\rho^{\frac{1}{2}} A^* \rho^{\frac{1}{2}} B)$$

Then, one can verify that $(\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2), \langle\langle \cdot, \cdot \rangle\rangle_{\omega_\rho})$ is the non-commutative Hilbert space (which can be called the quantum Liouville space). It will be denoted by $L_2(\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2), \rho)$. Moreover, one can show that \mathcal{L} is a well defined bounded Markov generator such that

$$\langle\langle \mathcal{L}(A), B \rangle\rangle_{\omega_\rho} = \langle\langle A, \mathcal{L}(B) \rangle\rangle_{\omega_\rho}$$

It easily follows that the following semigroup $T_t^\Lambda (\equiv T_t) = \exp(t\mathcal{L})$ is a well defined Markov uniformly continuous semigroup such that it is self-adjoint on the non-commutative Hilbert space, the state ω_ρ is invariant (with respect

to T_t) and T_t can be represented as the sum of the following convergent series:

$$\mathcal{I} + t\mathcal{L} + \frac{t^2}{2!}\mathcal{L}^2 + \dots$$

with \mathcal{I} being the identity operator.

3 Evolution of entanglement

In the next two sections we will look more closely at the time evolution of quantum correlations. Here, we will present the simplest nontrivial example clearly showing that quantum dynamics can produce this type of correlations. We will analyze a 2×2 system with block spin-flip dynamics. Thus, $\mathcal{H}_1 = \mathbb{C}^2 = \mathcal{H}_2$ and the Hilbert space of the composite system is given by $\mathbb{C}^2 \otimes \mathbb{C}^2 \approx \mathbb{C}^4$. Let $\{\xi_1, \xi_2\}$ be an orthonormal basis in \mathbb{C}^2 . We define:

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{2}}(\xi_1 \otimes \xi_1 + \xi_2 \otimes \xi_2) \\ x_2 &= \xi_1 \otimes \xi_2 \\ x_3 &= \xi_2 \otimes \xi_1 \\ x_4 &= \frac{1}{\sqrt{2}}(\xi_1 \otimes \xi_1 - \xi_2 \otimes \xi_2) \end{aligned} \tag{2}$$

One can easily check that $\{x_i\}_{i=1}^4$ forms the orthonormal basis in \mathbb{C}^4 . Let us define a faithful density matrix ρ on \mathbb{C}^4 which is given by the formula:

$$\rho := \sum_{i=1}^4 \lambda_i |x_i\rangle\langle x_i| \quad \lambda_i > 0, \quad \sum_{i=1}^4 \lambda_i = 1 \tag{3}$$

The just defined ρ will play the role of the reference state (cf. Introduction). We will need the following the well known fact which can be verified by straightforward calculation.

Proposition 3.1 *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. For every $x, v \in \mathcal{H}_1$ and $y, z \in \mathcal{H}_2$ we have: $Tr_1(|x \otimes y\rangle\langle v \otimes z|) = (x, v)|y\rangle\langle z|$, where Tr_1 is the partial trace over \mathcal{H}_1 in $\mathcal{H}_1 \otimes \mathcal{H}_2$ and (\cdot, \cdot) denotes the scalar product (in \mathcal{H}_1).*

In the sequel we shall identify $Tr_1(\cdot)$ with its embedding into $\mathcal{H}_1 \otimes \mathcal{H}_2$, defined as $\mathbf{1} \otimes Tr_1(\cdot)$. Now, let us consider the quantum spin-flip type dynamics T_t for our model. Its infinitesimal generator is defined as (cf. Section II):

$$\mathcal{L}(f) = E(f) - f$$

where $f \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, while E was defined in the previous section.

The dynamical semigroup T_t has the additional property: as $\mathcal{A} \subset L_2(\mathcal{A}, \rho)$, we have $T_t \mathcal{A} \subset \mathcal{A}$ (Feller property), where $\mathcal{A} = \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $L_2(\mathcal{A}, \rho)$ is the non-commutative Hilbert space.

It follows that the Feller property allows to study the following duality problem: We may consider the time evolution T_t as the family of maps $T_t : \mathcal{A} \rightarrow \mathcal{A}$, then we can apply the standard equivalence between Schrödinger and Heisenberg picture to determine the evolution T_t^d of a state σ . To this end we define:

$$Tr(T_t^d(\sigma)f) := Tr(\sigma T_t(f))$$

for any state σ and any observable f . Therefore, we are able to describe explicitly the time evolution of states for that type of dynamics, which is given by the following mapping:

$$\sigma \rightarrow T_t^d(\sigma) = \sigma + t(E^d(\sigma) - \sigma) + \dots \tag{4}$$

where the dual E^d of the infinitesimal generator E is defined by the equality:

$$Tr(E^d(\sigma)f) = Tr(\sigma E(f))$$

and the series in the right hand side of (4) is convergent. Using (1) we can write:

$$Tr(E^d(\sigma)f) = Tr(\sigma Tr_1(\gamma^* f \gamma)) = Tr(Tr_1(\sigma) \gamma^* f \gamma) = Tr(\gamma Tr_1(\sigma) \gamma^* f)$$

Thus, we get:

$$E^d(\sigma) = \gamma Tr_1(\sigma) \gamma^* = \rho^{\frac{1}{2}} (Tr_1 \rho)^{-\frac{1}{2}} Tr_1 \sigma (Tr_1 \rho)^{-\frac{1}{2}} \rho^{\frac{1}{2}} \tag{5}$$

Let us put $\sigma = \sigma^I \otimes \sigma^{II}$, and recall that ρ was defined by (3). Obviously

$$Tr_1 \sigma = \mathbf{1}_I \otimes \sigma^{II} \quad (6)$$

Using Proposition 3.1 we can easily calculate $Tr_1(\rho) = \sum_{i=1}^4 \lambda_i Tr_1 |x_i\rangle\langle x_i|$. We have

$$\begin{aligned} Tr_1 |x_1\rangle\langle x_1| &= \frac{1}{2} Tr_1 |\xi_1 \otimes \xi_1 + \xi_2 \otimes \xi_2\rangle\langle \xi_1 \otimes \xi_1 + \xi_2 \otimes \xi_2| = \\ &= \frac{1}{2} [Tr_1 |\xi_1 \otimes \xi_1\rangle\langle \xi_1 \otimes \xi_1| + Tr_1 |\xi_1 \otimes \xi_1\rangle\langle \xi_2 \otimes \xi_2| + \\ &+ Tr_1 |\xi_2 \otimes \xi_2\rangle\langle \xi_1 \otimes \xi_1| + Tr_1 |\xi_2 \otimes \xi_2\rangle\langle \xi_2 \otimes \xi_2|] = \\ &= \frac{1}{2} [\mathbf{1}_I \otimes (|\xi_1\rangle\langle \xi_1| + |\xi_2\rangle\langle \xi_2|)] = \frac{1}{2} [\mathbf{1}_{II}] \end{aligned}$$

For simplicity, we will denote $\mathbf{1}_I$ and $\mathbf{1}_{II}$ briefly by $\mathbf{1}$ when no confusion can arise. Analogously:

$$Tr_1 |x_2\rangle\langle x_2| = Tr_1 |\xi_1 \otimes \xi_2\rangle\langle \xi_1 \otimes \xi_2| = \mathbf{1} \otimes |\xi_2\rangle\langle \xi_2|$$

$$Tr_1 |x_3\rangle\langle x_3| = Tr_1 |\xi_2 \otimes \xi_1\rangle\langle \xi_2 \otimes \xi_1| = \mathbf{1} \otimes |\xi_1\rangle\langle \xi_1|$$

$$Tr_1 |x_4\rangle\langle x_4| = \frac{1}{2} Tr_1 |\xi_1 \otimes \xi_1 - \xi_2 \otimes \xi_2\rangle\langle \xi_1 \otimes \xi_1 - \xi_2 \otimes \xi_2| = \frac{1}{2} [\mathbf{1} \otimes \mathbf{1}]$$

Eventually, we obtain:

$$\begin{aligned} Tr_1 \rho &= \frac{\lambda_1}{2} \mathbf{1} + \lambda_2 \mathbf{1} \otimes |\xi_2\rangle\langle \xi_2| + \lambda_3 \mathbf{1} \otimes |\xi_1\rangle\langle \xi_1| + \frac{\lambda_4}{2} \mathbf{1} = \\ &= \left(\frac{\lambda_1 + \lambda_4}{2} + \lambda_3 \right) \mathbf{1} \otimes |\xi_1\rangle\langle \xi_1| + \left(\frac{\lambda_1 + \lambda_4}{2} + \lambda_2 \right) \mathbf{1} \otimes |\xi_2\rangle\langle \xi_2| \end{aligned} \quad (7)$$

We introduce the following notation:

$$\chi_1 = \left(\frac{\lambda_1 + \lambda_4}{2} + \lambda_3 \right)^{-1} \quad \chi_2 = \left(\frac{\lambda_1 + \lambda_4}{2} + \lambda_2 \right)^{-1}$$

Inserting (3), (6), and (7) into (5) we get:

$$\begin{aligned} E^d(\sigma) &= \left(\sum_{i=1}^4 \lambda_i^{\frac{1}{2}} |x_i\rangle\langle x_i| \right) \cdot \left(\mathbf{1} \otimes [\chi_1^{\frac{1}{2}} |\xi_1\rangle\langle \xi_1| + \chi_2^{\frac{1}{2}} |\xi_2\rangle\langle \xi_2|] \right) \cdot \\ &\cdot (\mathbf{1} \otimes \sigma^{II}) \cdot \left(\mathbf{1} \otimes [\chi_1^{\frac{1}{2}} |\xi_1\rangle\langle \xi_1| + \chi_2^{\frac{1}{2}} |\xi_2\rangle\langle \xi_2|] \right) \cdot \left(\sum_{i=1}^4 \lambda_i^{\frac{1}{2}} |x_i\rangle\langle x_i| \right) \end{aligned}$$

Now, suppose that $\sigma^{II} = a|\xi_1\rangle\langle \xi_1| + b|\xi_2\rangle\langle \xi_2|$ with $a \geq 0, b \geq 0, a + b = 1$. Then,

$$E^d(\sigma) = \left(\sum_{i=1}^4 \lambda_i^{\frac{1}{2}} |x_i\rangle\langle x_i| \right) \cdot (\mathbf{1} \otimes a\chi_1 |\xi_1\rangle\langle \xi_1| + \mathbf{1} \otimes b\chi_2 |\xi_2\rangle\langle \xi_2|) \cdot \left(\sum_{i=1}^4 \lambda_i^{\frac{1}{2}} |x_i\rangle\langle x_i| \right)$$

Using (2) and performing some lengthy calculation one can obtain:

$$\begin{aligned} E^d(\sigma^I \otimes (a|\xi_1\rangle\langle \xi_1| + b|\xi_2\rangle\langle \xi_2|)) &= \left[a\chi_1 (\lambda_1^{\frac{1}{2}} + \lambda_4^{\frac{1}{2}})^2 + b\chi_2 (\lambda_1^{\frac{1}{2}} - \lambda_4^{\frac{1}{2}})^2 \right] |\xi_1\rangle\langle \xi_1| \otimes |\xi_1\rangle\langle \xi_1| + \\ &+ (a\chi_1 + b\chi_2) (\lambda_1 - \lambda_4) |\xi_1\rangle\langle \xi_2| \otimes |\xi_1\rangle\langle \xi_2| + (a\chi_1 + b\chi_2) (\lambda_1 - \lambda_4) |\xi_2\rangle\langle \xi_1| \otimes |\xi_2\rangle\langle \xi_1| + \end{aligned}$$

$$\begin{aligned}
& + \left[a\chi_1(\lambda_1^{\frac{1}{2}} - \lambda_4^{\frac{1}{2}})^2 + b\chi_2(\lambda_1^{\frac{1}{2}} + \lambda_4^{\frac{1}{2}})^2 \right] |\xi_2\rangle\langle\xi_2| \otimes |\xi_2\rangle\langle\xi_2| + \\
& + a\chi_1\lambda_3|\xi_2\rangle\langle\xi_2| \otimes |\xi_1\rangle\langle\xi_1| + b\chi_2\lambda_2|\xi_1\rangle\langle\xi_1| \otimes |\xi_2\rangle\langle\xi_2|
\end{aligned}$$

Now, we assume that $\lambda_2 = \lambda_3$ which implies $\chi_1 = \chi_2 = \chi$. Then,

$$\begin{aligned}
E^d(\sigma^I \otimes (a|\xi_1\rangle\langle\xi_1| + b|\xi_2\rangle\langle\xi_2|)) & = \chi \left[a(\lambda_1^{\frac{1}{2}} + \lambda_4^{\frac{1}{2}})^2 + b(\lambda_1^{\frac{1}{2}} - \lambda_4^{\frac{1}{2}})^2 \right] |\xi_1\rangle\langle\xi_1| \otimes |\xi_1\rangle\langle\xi_1| + \\
& + (\lambda_1 - \lambda_4)|\xi_1\rangle\langle\xi_2| \otimes |\xi_1\rangle\langle\xi_2| + (\lambda_1 - \lambda_4)|\xi_2\rangle\langle\xi_1| \otimes |\xi_2\rangle\langle\xi_1| + \\
& + [a(\lambda_1^{\frac{1}{2}} - \lambda_4^{\frac{1}{2}})^2 + b(\lambda_1^{\frac{1}{2}} + \lambda_4^{\frac{1}{2}})^2] |\xi_2\rangle\langle\xi_2| \otimes |\xi_2\rangle\langle\xi_2| + \\
& \left[a\lambda_2|\xi_2\rangle\langle\xi_2| \otimes |\xi_1\rangle\langle\xi_1| + b\lambda_2|\xi_1\rangle\langle\xi_1| \otimes |\xi_2\rangle\langle\xi_2| \right]
\end{aligned}$$

Performing some easy but tedious calculations we arrive at the following decomposition:

$$E^d(\sigma) = \sum_{i=1}^4 \tilde{\lambda}_i |y_i\rangle\langle y_i|$$

with

$$\tilde{\lambda}_1 = \chi\lambda_+ \quad \tilde{\lambda}_2 = \chi b\lambda_2 \quad \tilde{\lambda}_3 = \chi a\lambda_2 \quad \tilde{\lambda}_4 = \chi\lambda_-$$

where

$$\lambda_{\pm} = \frac{A + C \pm X}{2} \quad X = \sqrt{(A - C)^2 + 4B^2}$$

and

$$A := a(\lambda_1^{\frac{1}{2}} + \lambda_4^{\frac{1}{2}})^2 + b(\lambda_1^{\frac{1}{2}} - \lambda_4^{\frac{1}{2}})^2 \quad B := \lambda_1 - \lambda_4 \quad C := a(\lambda_1^{\frac{1}{2}} - \lambda_4^{\frac{1}{2}})^2 + b(\lambda_1^{\frac{1}{2}} + \lambda_4^{\frac{1}{2}})^2$$

while $\{y_i\}_{i=1}^4$ is the orthonormal basis defined as below:

$$\begin{aligned}
y_1 & = \eta_+ \xi_1 \otimes \xi_1 + \kappa_+ \xi_2 \otimes \xi_2 \\
y_2 & = \xi_1 \otimes \xi_2 \\
y_3 & = \xi_2 \otimes \xi_1 \\
y_4 & = \eta_- \xi_1 \otimes \xi_1 + \kappa_- \xi_2 \otimes \xi_2
\end{aligned}$$

where

$$\eta_{\pm} := \frac{\sqrt{2}B}{\sqrt{X^2 \mp (A - C)X}} \quad \kappa_{\pm} := \frac{-(A - C) \pm X}{\sqrt{2}\sqrt{X^2 \mp (A - C)X}}$$

The above decomposition of $E^d(\sigma)$ is well defined for $\lambda_1 > \lambda_4$. In particular, we have $\lambda_{\pm} > 0$, $\eta_{\pm}, \kappa_{\pm} \neq 0$.

Now, we are in position to examine the separability of the state $E^d(\sigma^I \otimes (a|\xi_1\rangle\langle\xi_1| + b|\xi_2\rangle\langle\xi_2|))$. We will use the simple argument presented in [8]. Define:

$$E_0^d(\sigma) = \tilde{\lambda}_1 |y_1\rangle\langle y_1| + \tilde{\lambda}_4 |y_4\rangle\langle y_4|$$

We observe

$$E_0^d(\sigma)y_2 = 0 = E_0^d(\sigma)y_3$$

and

$$(E_0^d(\sigma)\xi_1 \otimes \xi_1, \xi_2 \otimes \xi_2) = \tilde{\lambda}_1 \eta_+ \kappa_+ + \tilde{\lambda}_4 \eta_- \kappa_- \equiv A$$

Let us put

$$S = \sum_j A_j \otimes B_j$$

where A_j (B_j) are positive operators in $\mathcal{B}(\mathcal{H}_1)$ ($\mathcal{B}(\mathcal{H}_2)$). Then,

$$\|E_0^d(\sigma) - S\| \geq A - \sum_j \|A_j^{\frac{1}{2}} \xi_1\| \|B_j^{\frac{1}{2}} \xi_2\| \|A_j^{\frac{1}{2}} \xi_2\| \|B_j^{\frac{1}{2}} \xi_1\|$$

Hence,

$$\|E_0^d(\sigma) - S\| \geq \frac{A}{2} > 0$$

Consequently, $E_0^d(\sigma)$ is not an element of the closure of separable states. To show that $(1-t)\sigma + tE^d(\sigma)$ is an entangled state we start with recalling some well known facts from the theory of partially ordered topological spaces. Let us consider $\mathcal{A}_{sa} = \{a \in \mathcal{A} : a = a^*\}$, where $\mathcal{A} = \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) = \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$ as a real finite dimensional Banach space (\mathcal{H}_1 and \mathcal{H}_2 were assumed to be finite dimensional Hilbert spaces). The set $\{conv(\mathcal{B}_+(\mathcal{H}_1) \otimes \mathcal{B}_+(\mathcal{H}_2))\}^{closure}$ will be denoted by V . Clearly, V is a proper generating cone in \mathcal{A} . Further, by general argument (or by direct proof) one can note that $\text{int}V \neq \emptyset$ (where int stands for interior of the set). Define $V_1 = V \cap K(0,1)$, where $K(0,1) = \{x \in \mathcal{A} : \|x\| \leq 1\}$. One can verify that V_1 is convex, compact subset with non-empty interior. Therefore, it is homeomorphic to unit ball, while its boundary ∂V_1 is homeomorphic to unit sphere. Thus, it follows that, in general, a convex combination of $\rho_1 \in V_1$ and $\rho_2 \notin V_1$ is not in V_1 . By applying the above facts to the case $\rho_1 = (1-t)\sigma$, $\rho_2 = tE^d(\sigma)$, and taking into account the equivalence of the trace and operator norms in finite dimensional case, one can draw the conclusion that within the perturbation calculus in the first order a separable state σ evolves to the entangled state $(1-t)\sigma + tE^d(\sigma)$.

4 Factorization of two-points correlation functions

In this section we will proceed with analysis of factorization of certain correlation functions. To clarify the relation of that topic to evolution of entanglement we start with establishing the relation between factorization and existence of non-quantum correlation between two subsystems $\mathcal{A}_1, \mathcal{A}_2$. Having this as well as the relation between quantum correlations and entanglement, we give a detailed analysis of factorization of two-point correlation functions.

Let us begin with a classical case, i.e. let \mathcal{A}_i , $i = 1, 2$, be two abelian C^* -algebras with identities. Consider $\mathcal{A}_{cl} = \mathcal{A}_1 \otimes \mathcal{A}_2$ and let $\omega : \mathcal{A}_{cl} \rightarrow \mathbb{C}$ be a state. The abelianness of \mathcal{A}_i , $i = 1, 2$, implies

$$\mathcal{A}_{cl} \cong \mathcal{C}(\Omega_1) \otimes \mathcal{C}(\Omega_2) = \mathcal{C}(\Omega_1 \times \Omega_2),$$

where Ω_i , $i = 1, 2$ are compact Hausdorff topological spaces. By Markov-Riesz theorem there exists a probability measure μ on $\Omega_1 \times \Omega_2$ such that

$$\omega(a_1 \otimes a_2) = \mu(\hat{a}_{12}),$$

with $\hat{a}_{12} = i(a_1 \otimes a_2)$ and i being an isomorphism between $\mathcal{C}(\Omega_1) \otimes \mathcal{C}(\Omega_2)$ and $\mathcal{C}(\Omega_1 \times \Omega_2)$. Consider the truncated correlation (or equivalently the second Ursell function)

$$C_{\omega, a_1, a_2} \equiv \omega(a_1 \otimes \mathbf{1} \cdot \mathbf{1} \otimes a_2) - \omega(a_1 \otimes \mathbf{1})\omega(\mathbf{1} \otimes a_2)$$

for $a_i \in \mathcal{A}_i$, $i = 1, 2$. Due to the fact that each (classical) measure can be $*$ -weakly approximated by finitely supported probability measures one has

$$C_{\omega, a_1, a_2} = \lim_n [\mu_n(\hat{a}_1 \cdot \hat{a}_2) - \mu_n(\hat{a}_1)\mu_n(\hat{a}_2)],$$

where for each n , $\mu_n = \sum_{i=1}^{N_n} \lambda_i^{(n)} \delta_{\omega_i^{(n)}}$, and $\delta_{\omega_i^{(n)}}$ is a Dirac (point) measure on $\Omega_1 \times \Omega_2$. Therefore $\delta_{\omega_i^{(n)}} = \delta_{\omega_i^{(n,1)}} \times \delta_{\omega_i^{(n,2)}}$, where we have used the following notation: $\Omega_1 \times \Omega_2 \ni \omega_i^n = (\omega_i^{(n,1)}, \omega_i^{(n,2)})$. Consequently

$$C_{\omega, a_1, a_2} = \lim_n \left[\sum_{i=1}^{N_n} \lambda_i^{(n)} \delta_{\omega_i^{(n,1)}}(\hat{a}_1) \delta_{\omega_i^{(n,2)}}(\hat{a}_2) - \left(\sum_{i=1}^{N_n} \lambda_i^{(n)} \delta_{\omega_i^{(n,1)}}(\hat{a}_1) \right) \left(\sum_{i=1}^{N_n} \lambda_i^{(n)} \delta_{\omega_i^{(n,2)}}(\hat{a}_2) \right) \right].$$

Thus, each classical truncated correlation C_{ω, a_1, a_2} can be approximated by a difference of "separable states" and product of one-point functions. The important point to note here is that from the very beginning it is necessary to determine subalgebras (here $\mathcal{A}_1, \mathcal{A}_2$) - then we can define correlations with respect to this fixed partition. We recall that we are studying block spin-flip dynamics, where the spin-flip is carried out over the region Λ_I . Thus, the partition was fixed. Moreover, by definition, a separable state has the factorization property!

Now, let \mathcal{A}_i be arbitrary (non-commutative) C^* -algebras with identities. Let ω be a separable state on $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, where in our case $\mathcal{A} = \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$, $\mathcal{A}_i = \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$. Then, guided by the above general observation

concerning classical systems, we say that a separable state $\omega = \sum_i \lambda_i \rho_i^I \otimes \rho_i^{II}$ encodes classical correlations with respect to the partition of $\mathcal{A} = \langle \mathcal{A}_1 \otimes \mathbf{1}, \mathbf{1} \otimes \mathcal{A}_2 \rangle$, a C^* -algebra generated by $\mathcal{A}_1 \otimes \mathbf{1}$ and $\mathbf{1} \otimes \mathcal{A}_2$.

Again, let ω be a separable state with respect to the partition $\mathcal{A} = \langle \mathcal{A}_1 \otimes \mathbf{1}, \mathbf{1} \otimes \mathcal{A}_2 \rangle$ and $T_t : \mathcal{A} \rightarrow \mathcal{A}$ be a spin-flip type dynamics. We wish to consider the truncated correlation (now for our quantum dynamical system)

$$C_{\omega,g,f}^{q,T} = \omega(gT_t(f)) - \omega(g)\omega(T_t(f))$$

with $g \in id_{\mathcal{H}_1} \otimes \mathcal{A}_2$, $f \in \mathcal{A}_1 \otimes id_{\mathcal{H}_2}$. Throughout the rest of the paper we shall assume that observables f and g are of the form:

$$f = F \otimes id_{\mathcal{H}_2}, \quad g = id_{\mathcal{H}_1} \otimes G$$

with $F \in \mathcal{B}(\mathcal{H}_1)$ and $G \in \mathcal{B}(\mathcal{H}_2)$, i.e. f (g) is an element of the subsystem I (II respectively). We observe:

$$\begin{aligned} C_{\omega,g,f}^{q,t} &= \omega(gT_t(f)) - \omega(g)\omega(T_t(f)) = \omega(gf) + t\omega(g\mathcal{L}(f)) + \frac{t^2}{2!}\omega(g\mathcal{L}(\mathcal{L}(f))) + \dots - \omega(g)\omega(f) \\ &= C_{\omega,g,f}^q + tC_{\omega,g,f}^{L,1} + \frac{t^2}{2!}C_{\omega,g,f}^{L,2} + \frac{t^3}{3!}C_{\omega,g,f}^{L,3} + \dots \end{aligned}$$

where

$$C_{\omega,g,f}^q = \omega(gf) - \omega(g)\omega(f)$$

$$C_{\omega,g,f}^{L,1} = \omega(g\mathcal{L}(f)) - \omega(g)\omega(\mathcal{L}(f)), \quad C_{\omega,g,f}^{L,2} = \omega(g\mathcal{L}(\mathcal{L}(f))) - \omega(g)\omega(\mathcal{L}(\mathcal{L}(f))), \quad \dots$$

We observe:

i) $C_{\omega,g,f}^q$ measures the "classical" correlations between g and f with respect to the partition $\mathcal{A}_1, \mathcal{A}_2$, for time equal to 0.

ii) We can not say the same for $C_{\omega,g,f}^{L,1}, C_{\omega,g,f}^{L,2}$, etc., since ω is separable with respect to the partition $\mathcal{A}_1 \otimes \mathbf{1}, \mathbf{1} \otimes \mathcal{A}_2$, while $g \in \mathcal{A}_2$ but $\mathcal{L}(f) \in \mathcal{A}$! We would have "classical" correlations if ω was of the form $\omega = \sum_i \lambda_i \omega_i^I \otimes \omega_i^{II}$, where ω_i^I is a state on a C^* -subalgebra containing $\mathcal{L}(f)$ while ω_i^{II} is a state on \mathcal{A}_2 , and similarly for higher order terms. In other words, the evolution (its infinitesimal generator) leads to a deviation from the original partition of the composite system and the new partition does not fit to our definition of the spin-flip operation over the region I . In particular, the given separable decomposition of ω is not adapted to be a measure of classical correlations between g and $\mathcal{L}(f)$.

Therefore, nonzero value of $\omega(G\mathcal{L}(F)) \equiv C_{\omega,g,f}^{L,1}$ can be taken as an indicator for the increment of non-classical correlations (with respect to the partition $\langle \mathcal{A}_1 \otimes \mathbf{1}, \mathbf{1} \otimes \mathcal{A}_2 \rangle$).

To elaborate that point a little bit further let the state ω be given by the density matrix ρ of the form

$$\rho = \sum_i \lambda_i \rho_i^I \otimes \rho_i^{II}. \quad (8)$$

with ρ_i^I and ρ_i^{II} being the states of the subsystem I and II , respectively. Consider $\omega(GE(F)) = \sum_i \lambda_i Tr(\rho_i^I \otimes \rho_i^{II} \cdot G \cdot E(F))$ with $G \in \mathcal{A}_2$, $F \in \mathcal{A}_1$ (with a natural embedding of \mathcal{A}_1 and \mathcal{A}_2 into \mathcal{A}). Assume $G = HH^*$ with $[H, \rho_i^{II}] = 0$. Then,

$$\begin{aligned} \omega(GE(F)) &= \sum_i \lambda_i Tr \rho_i^I \otimes H \rho_i^{II} H^* E(F) = \sum_i \lambda_i \rho_i^{II}(HH^*) Tr \rho_i^I \otimes \rho_i^{II,H} E(F) = \\ &= Tr E^d \left(\sum_i \lambda_i \rho_i^{II}(HH^*) \rho_i^I \otimes \rho_i^{II,H} \right) \cdot F \end{aligned}$$

with $\rho_i^{II,H} = \rho_i^{II} / \rho_i(H^*H)$. We know, by Section III, that $E^d \left(\sum_i \lambda_i \rho_i^{II}(HH^*) \rho_i^I \otimes \rho_i^{II,H} \right)$ is not, in general, a separable state. In other words, any reasonable factorization does not hold. In particular,

$$E^d \left(\sum_i \lambda_i \rho_i^{II}(HH^*) \rho_i^I \otimes \rho_i^{II,H} \right) (F \otimes id_{\mathcal{H}_2}) \neq \sum_i \lambda_i \rho_i^{II}(HH^*) \rho_i^I(F)$$

We can summarize the above consideration as follows:

An entangled state encodes non-classical correlations. Even in the simplest case of 2×2 system the spin-flip dynamics can lead to entangled state (cf. Section III). Moreover, in general, for the entangled state, the second Ursell function fails to have the factorization property.

Before performing the promised analysis we want to make an additional observation, which justifies our assumption that it is enough to consider only observables from subalgebras while considering correlation function $\omega(GE(F))$.

Remark 4.1 We note that the equality

$$0 = \omega(G\mathcal{L}(F)) = \omega(GE(F)) - \omega(GF) \quad (9)$$

for all $G \in \mathcal{A}$ and $F \in \mathcal{A}_2$ is a very strong condition, since it implies $E = id$, so the spin-flip dynamics would be a trivial one. Therefore we restrict ourselves to study much weaker condition

$$\omega(GE(F)) \neq \omega(GF)$$

for $G \in \mathcal{A}_1$, $F \in \mathcal{A}_2$ and a separable state ω on $\mathcal{A}_1 \otimes \mathcal{A}_2$.

Consequently, we shall analyze conditions under which the function $\langle G, F \rangle \rightarrow \omega(GE(F))$ does not factorize, i.e. (9) does not hold for $G \in \mathcal{A}_1$, $F \in \mathcal{A}_2$. Again, we assume that the state ρ is of the form (8). From now on we make the assumption that the density matrix ρ is an invertible one. This assumption stems from the general strategy of constructing quantum maps. Namely, we associate with the quantum system the quantum Hilbert space $L_2(\mathcal{A}, \rho)$ with a given reference Gibbs state. Obviously, any Gibbs state has the assumed property. We also note that if ρ is an invertible density matrix, then $Tr_1(\rho)$ has this property too. This observation will be used throughout this section.

As a result of longish (however not difficult) calculations we get the following characterization of the considered correlation functions:

Let $\{\varphi_i \otimes \phi_j\}$ be the orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$, where $\{\varphi_i\}$ is arbitrary orthonormal basis of \mathcal{H}_1 while the basis $\{\phi_j\}$ in \mathcal{H}_2 is such that $Tr_1(\rho)$ is diagonal. Then we have:

$$\langle E(f)g \rangle_\rho = \sum_{klji} \left(\sum_{pq} \rho_{pkjq}^{\frac{1}{2}} \rho_{lpqi}^{\frac{1}{2}} \sqrt{\frac{\tilde{a}_j}{\tilde{a}_i}} \right) \langle \varphi_k | F | \varphi_l \rangle \langle \phi_i | G | \phi_j \rangle \quad (10)$$

where

$$\tilde{a}_j = \sum_r \rho_{rrjj}$$

and the matrix elements in the basis $\{\varphi_i \otimes \phi_j\}$ are:

$$\rho_{pqrs} := \langle \varphi_p \otimes \phi_r | \rho | \varphi_q \otimes \phi_s \rangle \quad \rho_{pqrs}^{\frac{1}{2}} := \langle \varphi_p \otimes \phi_r | \rho^{\frac{1}{2}} | \varphi_q \otimes \phi_s \rangle \quad (11)$$

Moreover, \tilde{a}_j are the eigenvalues of $Tr_1(\rho)$.

4.1 General characterization of factorization

In order to examine the just described correlation functions, we assume as before that ρ , ρ^I and ρ^{II} (with or without indexes) denote states on $\mathcal{H}_1 \otimes \mathcal{H}_2$, \mathcal{H}_1 and \mathcal{H}_2 , respectively.

It is evident from the definition of $C_{\omega, f, g}^{L, 1}$ that the factorization of the correlation function $\langle E(f)g \rangle_\rho$ is not dependent on a particular choice of the decomposition of ρ . This allows us to examine the factorization for any decomposition of ρ with conclusions valid for any other decomposition. As the next step of mathematical framework for characterizing of correlation functions, we want to give the necessary and sufficient condition for the factorization of such functions.

Proposition 4.1 Let $\rho = \sum_i \lambda_i \rho_i^I \otimes \rho_i^{II}$ and $f = F \otimes id_{\mathcal{H}_2}$, $g = id_{\mathcal{H}_1} \otimes G$, $dim \mathcal{H}_1 = n$, $dim \mathcal{H}_2 = m$. Let $\{\varphi_i\}_{i=1}^n$ be arbitrary orthonormal basis of \mathcal{H}_1 and $\{\phi_j\}_{j=1}^m$ be orthonormal basis of \mathcal{H}_2 such that $\tilde{\rho} \equiv Tr_1 \rho$ is diagonal. Then the following conditions are equivalent:

(i) correlation function can be factorized, i.e.

$$\langle E(f)g \rangle_\rho = \sum_i \lambda_i \langle F \rangle_{\rho_i^I} \langle G \rangle_{\rho_i^{II}}$$

(ii) for every $k, l \in \{1, 2, \dots, n\}$ and $j, i \in \{1, 2, \dots, m\}$ the following equality holds:

$$\rho_{lkji} = \sum_{p=1}^n \sum_{q=1}^m \rho_{pkjq}^{\frac{1}{2}} \rho_{lpqi}^{\frac{1}{2}} \sqrt{\frac{\sum_{r=1}^n \rho_{rrjj}}{\sum_{v=1}^n \rho_{vvii}}} \quad (12)$$

where $\rho_{lkji} := \langle \varphi_l \otimes \phi_j | \rho | \varphi_k \otimes \phi_i \rangle$.

If the condition (ii) holds for some basis $\{\varphi_i\}$ then it holds also for any other basis $\{\tilde{\varphi}_i\}$ (basis $\{\phi_j\}$ unchanged!). Conversely, if (ii) does not hold for given basis $\{\varphi_i\}$ then it does not hold for any other basis $\{\tilde{\varphi}_i\}$.

Proof. Take the bases $\{\varphi_k\}$, $\{\phi_j\}$ such as described in the proposition. Calculate $\sum_i \lambda_i \langle F \rangle_{\rho_i^I} \langle G \rangle_{\rho_i^{II}}$ in the basis $\{\varphi_k \otimes \phi_j\}$. Using (11), we get:

$$\sum_i \lambda_i \langle F \rangle_{\rho_i^I} \langle G \rangle_{\rho_i^{II}} = \sum_i \lambda_i \sum_l \langle \varphi_l | \rho_i^I F | \varphi_l \rangle \sum_j \langle \phi_j | \rho_i^{II} G | \phi_j \rangle = \sum_{kljr} \rho_{lkjr} \langle \varphi_k | F | \varphi_l \rangle \langle \phi_r | G | \phi_j \rangle$$

From the general form of the considered correlation function (cf. (10)) we have:

$$\langle E(f)g \rangle_\rho = \sum_{kljr} \left(\sum_{pq} \rho_{pkjq}^{\frac{1}{2}} \rho_{lpqr}^{\frac{1}{2}} \sqrt{\frac{\tilde{a}_j}{\tilde{a}_r}} \right) \langle \varphi_k | F | \varphi_l \rangle \langle \phi_r | G | \phi_j \rangle \quad \text{where} \quad \tilde{a}_j = \sum_v \rho_{vvjj}$$

Substituting $F = |\varphi_k\rangle\langle\varphi_l|$ and $G = |\phi_j\rangle\langle\phi_i|$ with k, l, j, i arbitrary, it is easy to verify that the right-hand sides of the last two expressions equal if and only if (12) holds.

4.2 Factorization and quasi-classicality

It turns out that the sufficient conditions for factorization of correlation function can be connected to the 'quasi-classicality' of the considered state - the notion to be precised in the following definition. If $\rho = \sum_i \lambda_i \rho_i^I \otimes \rho_i^{II}$ is given decomposition of a separable density matrix, then ρ can be considered as classical if $\{\rho_i^I\}$ and $\{\rho_i^{II}\}$ are abelian families of density matrices. Below, we define weaker conditions for families of density matrices, which are essential for the subsequent considerations.

Definition 4.1 Let \mathcal{H} be the Hilbert space, $\dim \mathcal{H} = n$. Let $\{\rho_i\}_{i=1}^N$ be a family of density matrices on \mathcal{H} and $\{\lambda_i\}_{i=1}^N$ be (strictly) positive numbers ($\lambda_i > 0$), such that $\sum_{i=1}^N \lambda_i = 1$. Define $\rho_0 := \sum_i \lambda_i \rho_i$. Let $\{\phi_j\}_{j=1}^n$ - orthonormal basis consisting of eigenvectors of ρ_0 and $\{c_j\}_{j=1}^n$ - corresponding eigenvalues. We will say that the family $\{\rho_i\}$ is K -quasi-abelian ($1 \leq K \leq n$) if and only if the following condition holds: there exists the partition $\{A_p\}_{p=1}^K$ of $\{1, 2, \dots, n\}$, $|A_p| \geq 1$ (where $|A|$ denotes the cardinality of the set A), such that:
(i) for any $p \in \{1, \dots, K\}$ and $r, s \in A_p$ we have $c_r = c_s$ and for any $r \in A_p$, $s \in A_q$ if $p \neq q$ then $c_r \neq c_s$
(ii) for any $k, l \in \{1, \dots, n\}$ and $i \in \{1, \dots, N\}$ if $c_k \neq c_l$ then $\langle \phi_k | \rho_i | \phi_l \rangle = 0$

Remark 4.2 Definition 4.1 can be rephrased in a simple way as follows. The family $\{\rho_i\}$ is K -quasi-abelian if and only if ρ_i commutes with spectral projectors of ρ_0 (so with ρ_0 itself) for each $i = 1, \dots, N$. However, we shall use properties (i) and (ii) in the sequel, so the original formulation is more convenient.

It is an easy observation that if $|A_p| = 1$ for some p , then the corresponding vector ϕ_p is the common eigenvector for all the matrices ρ_i . This means that the number of sets A_p in the partition $\{A_p\}$ with the cardinality one equals the number of common eigenvectors for all the matrices ρ_i . In particular, if $\{\rho_i\}$ is K -quasi-abelian for $K = n$, then it is abelian in the traditional sense.

On the basis of the above definition we can formulate the sufficient conditions for factorization of the correlation function using the above type of commutativity properties of the families $\{\rho_i^I\}$ and $\{\rho_i^{II}\}$.

Proposition 4.2 Let $\rho = \sum_i \lambda_i \rho_i^I \otimes \rho_i^{II}$ be a separable density matrix and $f = F \otimes id_{\mathcal{H}_2}$, $g = id_{\mathcal{H}_1} \otimes G$, $\dim \mathcal{H}_1 = n$, $\dim \mathcal{H}_2 = m$. Then the following implication holds:

$$\left(\begin{array}{l} \text{There exists decomposition } \rho = \sum_{j=1}^{\tilde{N}} \tilde{\lambda}_j \tilde{\rho}_j^I \otimes \tilde{\rho}_j^{II} \text{ such that one of the conditions is satisfied:} \\ \text{(i) } \{\tilde{\rho}_j^{II}\} \text{ is } K\text{-quasi-abelian } (K < n) \text{ and } \{\tilde{\rho}_j^I\} \text{ is abelian} \\ \text{(ii) } \{\tilde{\rho}_j^{II}\} \text{ is abelian} \end{array} \right)$$

$$\text{then } \left(\langle E(f)g \rangle_\rho = \sum_i \lambda_i \langle F \rangle_{\rho_i^I} \langle G \rangle_{\rho_i^{II}} \text{ (factorization of the correlation function)} \right)$$

Proof. We will prove the implication assuming (i). One can prove the statement under (ii) by similar reasoning. Let us first prove that (12) holds. To this end let us take the bases $\{\varphi_i\}$ and $\{\phi_j\}$ such that $\tilde{\rho}_j^I$ are diagonal in the basis $\{\varphi_i\}$ and $\sum_j \tilde{\lambda}_j \tilde{\rho}_j^{II} \equiv Tr_1(\rho)$ is diagonal in the basis $\{\phi_j\}$. Then, the matrix elements $\rho_{lkij} = 0$ whenever $l \neq k$. The same is true for $\rho_{lkij}^{\frac{1}{2}}$. Note that also $\sum_{pq} \rho_{pkjq}^{\frac{1}{2}} \rho_{lpqi}^{\frac{1}{2}} \sqrt{\tilde{a}_j / \tilde{a}_i} = 0$ for $l \neq k$ since for every p, i, j and q either $\rho_{pkjq}^{\frac{1}{2}} = 0$ or $\rho_{lpqi}^{\frac{1}{2}} = 0$. Hence, for $l \neq k$ (12) holds. Now, let $l = k$. Writing $\sum_{pq} \rho_{pkjq}^{\frac{1}{2}} \rho_{lpqi}^{\frac{1}{2}} = \sum_p (\sum_q \rho_{pkjq}^{\frac{1}{2}} \rho_{lpqi}^{\frac{1}{2}})$, we see that it can differ from zero if and only if $p = k (= l)$. Taking into account that due to our specific choice of the bases, the equality $\sum_q \rho_{kkjq}^{\frac{1}{2}} \rho_{kqii}^{\frac{1}{2}} = \rho_{kkji}$ holds, we can write (12) for $l = k$ in the form $\rho_{kkji} = \rho_{kkji} \sqrt{\tilde{\rho}_{jj} / \tilde{\rho}_{ii}}$ with $\tilde{\rho}_{jj} := \langle \phi_j | \tilde{\rho} | \phi_j \rangle$, $\tilde{\rho} = \sum_s \tilde{\lambda}_s \tilde{\rho}_s^{II} (\equiv Tr_1(\rho))$. Note that because of the choice of $\{\phi_j\}$, elements $\tilde{\rho}_{jj}$ are the

eigenvalues of $\tilde{\rho}$. By assumption, the family $\{\tilde{\rho}_j^{II}\}$ is K-quasi-abelian. This means that either $\tilde{\rho}_{jj} = \tilde{\rho}_{ii}$ ((12) is then satisfied in an obvious way) or $\tilde{\rho}_{jj} \neq \tilde{\rho}_{ii}$. In the latter case we have $\langle \phi_j | \tilde{\rho}_s^{II} | \phi_i \rangle = 0$ (cf. Def. 4.1), which implies $\rho_{kkji} = \sum_s \tilde{\lambda}_s \langle \varphi_k | \tilde{\rho}_s^I | \varphi_k \rangle \langle \phi_j | \tilde{\rho}_s^{II} | \phi_i \rangle = 0$ and, of course, (12) holds. We have shown that (i) implies 0(12). By Proposition 4.1 we have $\langle E(f)g \rangle_\rho = \sum_j \tilde{\lambda}_j \langle F \rangle_{\tilde{\rho}_j^I} \langle G \rangle_{\tilde{\rho}_j^{II}}$ and thus, by equivalence of the decompositions of ρ , we get $\langle E(f)g \rangle_\rho = \sum_i \lambda_i \langle F \rangle_{\rho_i^I} \langle G \rangle_{\rho_i^{II}}$ which ends the proof.

The implication in Proposition 4.2 can be partially inverted. We have

Proposition 4.3 *Let $\rho = \sum_i \lambda_i \rho_i^I \otimes \rho_i^{II}$, $f = F \otimes id_{\mathcal{H}_2}$, $g = id_{\mathcal{H}_1} \otimes G$, $\dim \mathcal{H}_1 = n$, $\dim \mathcal{H}_2 = m$. Assume that there exists decomposition $\rho = \sum_j \tilde{\lambda}_j \tilde{\rho}_j^I \otimes \tilde{\rho}_j^{II}$ such that $\{\tilde{\rho}_j^I\}$ is abelian. Then the following conditions are equivalent:*

(i) *For some decomposition $\rho = \sum_j \hat{\lambda}_j \hat{\rho}_j^I \otimes \hat{\rho}_j^{II}$ we have:*

$$\{\hat{\rho}_j^I\} \text{ is abelian and } \{\hat{\rho}_j^{II}\} \text{ is } K\text{-quasi-abelian } (K \leq m)$$

$$(ii) \langle E(f)g \rangle_\rho = \sum_i \lambda_i \langle F \rangle_{\rho_i^I} \langle G \rangle_{\rho_i^{II}}$$

Proof. ((i) \Rightarrow (ii)) This implication follows from Proposition 4.2.

((ii) \Rightarrow (i)) Let $\{\varphi_i\}$ be the orthonormal basis of \mathcal{H}_1 , in which $\tilde{\rho}_j^I$ are diagonal and $\{\phi_j\}$ be the orthonormal basis consisting of eigenvectors of the reduced density matrix $\tilde{\rho} = Tr_1(\rho)$. Then there exists decomposition $\rho = \sum_j \hat{\lambda}_j \hat{\rho}_j^I \otimes \hat{\rho}_j^{II}$ such that $\hat{\rho}_j^I = |\varphi_j\rangle\langle\varphi_j|$ (the family $\{\hat{\rho}_j^I\}$ is abelian). The matrix elements of ρ are $\rho_{kluw} = \sum_i \lambda_i \eta_{ikl} \xi_{iuw}$ with $\eta_{ikl} = \langle \varphi_k | \hat{\rho}_i^I | \varphi_l \rangle = \delta_{ik} \delta_{il}$ and $\xi_{iuw} = \langle \phi_u | \hat{\rho}_i^{II} | \phi_w \rangle$. Obviously, $\langle E(f)g \rangle_\rho = \sum_i \lambda_i \langle F \rangle_{\rho_i^I} \langle G \rangle_{\rho_i^{II}}$ implies $\langle E(f)g \rangle_\rho = \sum_j \hat{\lambda}_j \langle F \rangle_{\hat{\rho}_j^I} \langle G \rangle_{\hat{\rho}_j^{II}}$. By Proposition 4.1 it means that

$$\rho_{lkji} = \sum_{p=1}^n \sum_{q=1}^m \rho_{pkjq}^{\frac{1}{2}} \rho_{lpqi}^{\frac{1}{2}} \sqrt{\frac{\tilde{\rho}_{jj}}{\tilde{\rho}_{ii}}} \quad (13)$$

where $\tilde{\rho}_{jj}$ is eigenvalue of $\tilde{\rho}$ corresponding to ϕ_j . Since $\rho_{klji} = 0$ if $l \neq k$ (which implies $\rho_{klji}^{\frac{1}{2}} = 0$ for $l \neq k$), (13) reduces to $\rho_{kkji} = \sum_{q=1}^m \rho_{kkjq}^{\frac{1}{2}} \rho_{kkqi}^{\frac{1}{2}} \sqrt{\tilde{\rho}_{jj}/\tilde{\rho}_{ii}}$, and $\sum_q \rho_{kkjq}^{\frac{1}{2}} \rho_{kkqi}^{\frac{1}{2}} = \rho_{kkji}$. Hence, if (13) holds, then for every $j, i \in \{1, \dots, m\}$ such that $j \neq i$ we have: if $\tilde{\rho}_{jj} \neq \tilde{\rho}_{ii}$ then $\rho_{kkji} = 0$ for $k = 1, \dots, n$. But $\rho_{kkji} = 0$ means $\sum_l \lambda_l \eta_{lkk} \xi_{lji} = 0$. Since $\eta_{lkk} = \delta_{lk} \delta_{il}$, we have $0 = \sum_l \lambda_l \eta_{lkk} \xi_{lji} = \sum_l \lambda_l \delta_{lk} \xi_{lji} = \lambda_k \xi_{kji} \Rightarrow \xi_{kji} = 0$ as from the definition of decomposition $\forall_l \lambda_l > 0$. Thus, the set $\{\hat{\rho}_j^{II}\}$ has the following property: if $\tilde{\rho}_{jj} \neq \tilde{\rho}_{ii}$ then for every $k \in \{1, \dots, n\}$ we have $(\hat{\rho}_k^{II})_{ji} \equiv \xi_{kji} = 0$. From Definition 4.1 it follows that the set $\{\rho_i^{II}\}$ is K-quasi-abelian for some $K \leq m$.

4.3 Factorization and nondegeneracy of density matrix

The next result provides a criterion for the factorization of correlation functions under the nondegeneracy condition specified below. To show this equivalence we need the following result:

Lemma 4.1 *Let $\rho = \sum_i \lambda_i \rho_i^I \otimes \rho_i^{II}$, $\dim \mathcal{H}_1 = n$, $\dim \mathcal{H}_2 = m$ and the matrix elements ρ_{klji} satisfy in some product basis $\{\varphi_i \otimes \phi_j\}$ the following condition $\rho_{klji} = 0$ whenever $k \neq l$ ($\rho_{klji} = 0$ whenever $j \neq i$). Then there exists decomposition $\rho = \sum_j \tilde{\lambda}_j \tilde{\rho}_j^I \otimes \tilde{\rho}_j^{II}$ such that $\{\tilde{\rho}_j^I\}$ is abelian ($\{\tilde{\rho}_j^{II}\}$ is abelian).*

Proof. Let us consider the matrix representation of ρ , i.e. $\rho = [\rho_{klji}]_{k,l=1,\dots,n; j,i=1,\dots,m}$ (cf. (11)). Suppose that $\rho_{klji} = 0$ whenever $k \neq l$. Let $\tilde{\lambda}_s := \sum_{p=1}^m \rho_{sspp}$. Consider the following decomposition of ρ :

$$\rho = \sum_{s=1}^n \tilde{\lambda}_s \tilde{\rho}_s^I \otimes \tilde{\rho}_s^{II} \quad (14)$$

where

$$\tilde{\rho}_s^I = |\varphi_s\rangle\langle\varphi_s| \quad [\tilde{\rho}_s^{II}]_{ji} = \begin{cases} \frac{1}{\tilde{\lambda}_s} \cdot [\rho_{ssji}]_{j,i=1}^m & \text{if } \tilde{\lambda}_s > 0 \\ 0 & \text{if } \tilde{\lambda}_s = 0 \end{cases}$$

One can easily check that (14) is a well defined decomposition of ρ (in particular if $\tilde{\lambda}_s = 0$ then $\rho_{ssji} = 0$ for $j, i = 1, \dots, m$). Of course $\{\tilde{\rho}_s^I\}$ is abelian. The proof of the second statement is similar.

Now we are in position to give the promised result which shows a relation between factorization of correlation function and the spectral properties (nondegeneracy) of density matrix.

Proposition 4.4 Let $\rho = \sum_i \lambda_i \rho_i^I \otimes \rho_i^{II}$, $f = F \otimes id_{\mathcal{H}_2}$, $g = id_{\mathcal{H}_1} \otimes G$, $\dim \mathcal{H}_1 = n$, $\dim \mathcal{H}_2 = m$. Assume that the reduced density matrix $\tilde{\rho} = \sum_i \lambda_i \rho_i^{II} (\equiv Tr_1(\rho))$ has nondegenerated eigenvalues. Then the following conditions are equivalent:

- (i) there exists decomposition $\rho = \sum_j \tilde{\lambda}_j \tilde{\rho}_j^I \otimes \tilde{\rho}_j^{II}$ such that $\{\tilde{\rho}_j^{II}\}$ is abelian
- (ii) correlation function can be factorized, i.e. $\langle E(f)g \rangle_\rho = \sum_i \lambda_i \langle F \rangle_{\rho_i^I} \langle G \rangle_{\rho_i^{II}}$

Proof. ((i) \Rightarrow (ii)) This implication follows from Proposition 4.2.

((ii) \Rightarrow (i)) Let $\{\varphi_i\}_{i=1}^n$ be arbitrary orthonormal basis of \mathcal{H}_1 and $\{\phi_j\}_{j=1}^m$ be orthonormal basis of \mathcal{H}_2 consisting of eigenvectors of $\tilde{\rho}$. If the correlation function factorizes, then from Proposition 4.1 we have $\rho_{lkji} = \sum_{p=1}^n \sum_{q=1}^m \rho_{pkjq}^{\frac{1}{2}} \rho_{lpqi}^{\frac{1}{2}} \sqrt{\tilde{\rho}_{jj}/\tilde{\rho}_{ii}}$ with $\tilde{\rho}_{ii}$ - the eigenvalue of $\tilde{\rho}$ corresponding to ϕ_i . This equality and self-adjointness of ρ imply:

$$\sum_{pq} \rho_{pkjq}^{\frac{1}{2}} \rho_{lpqi}^{\frac{1}{2}} \sqrt{\frac{\tilde{\rho}_{jj}}{\tilde{\rho}_{ii}}} = \sum_{rs} \rho_{rlis}^{\frac{1}{2}} \rho_{krsj}^{\frac{1}{2}} \sqrt{\frac{\tilde{\rho}_{ii}}{\tilde{\rho}_{jj}}} = \sum_{rs} \rho_{rlis}^{\frac{1}{2}} \rho_{krsj}^{\frac{1}{2}} \sqrt{\frac{\tilde{\rho}_{ii}}{\tilde{\rho}_{jj}}} = \sum_{rs} \rho_{rkjs}^{\frac{1}{2}} \rho_{lr si}^{\frac{1}{2}} \sqrt{\frac{\tilde{\rho}_{ii}}{\tilde{\rho}_{jj}}}$$

Multiplying the above equality by $\left(\frac{\tilde{\rho}_{jj}}{\tilde{\rho}_{ii}}\right)$ gives:

$$\underbrace{\sum_{pq} \rho_{pkjq}^{\frac{1}{2}} \rho_{lpqi}^{\frac{1}{2}} \sqrt{\frac{\tilde{\rho}_{jj}}{\tilde{\rho}_{ii}}}}_{\equiv \rho_{lkji}} \cdot \frac{\tilde{\rho}_{jj}}{\tilde{\rho}_{ii}} = \underbrace{\sum_{rs} \rho_{rkjs}^{\frac{1}{2}} \rho_{lr si}^{\frac{1}{2}} \sqrt{\frac{\tilde{\rho}_{jj}}{\tilde{\rho}_{ii}}}}_{\equiv \rho_{lkji}} \Rightarrow \rho_{lkji} \left(\frac{\tilde{\rho}_{jj}}{\tilde{\rho}_{ii}}\right) = \rho_{lkji} \Rightarrow \rho_{lkji}(\tilde{\rho}_{jj} - \tilde{\rho}_{ii}) = 0$$

By the assumption $\tilde{\rho}_{jj} \neq \tilde{\rho}_{ii}$, hence $\rho_{lkji} = 0$ whenever $j \neq i$. Our Proposition follows then from Lemma 4.1.

The results of this section provide a natural and intrinsic characterization of the two-points correlation function for block spin-flip dynamics and for the initial separable state. But one question still unanswered is whether the factorization or non-factorization of such functions is a genuine property for the considered dynamics. To answer this question we want to show that there are a lot of separable density matrices for which the correlation function $\langle E(f)g \rangle_\rho$ can not be factorized. Namely, we have the following:

Proposition 4.5 The set of density matrices such that the equality $\langle E(f)g \rangle_\rho = \sum_i \lambda_i \langle F \rangle_{\rho_i^I} \langle G \rangle_{\rho_i^{II}}$ does not hold, is dense in \tilde{S}_{sep} where $\tilde{S}_{sep} = \{\rho \in S_{sep} : Tr_1 \rho \text{ is invertible}\}$.

The proof of Proposition 4.5 is given in Appendix B.

We want to complete this section with the observation that there exists a strict connection between the problem of factorization of the correlation function and the separability of the square root $\rho^{\frac{1}{2}}$. Namely:

Proposition 4.6 Let $\rho = \sum_i \lambda_i \rho_i^I \otimes \rho_i^{II}$. If there exists decomposition $\sum_j \tilde{\lambda}_j \tilde{\rho}_j^I \otimes \tilde{\rho}_j^{II}$ such that one of the following conditions is satisfied:

- (i) $\{\tilde{\rho}_j^I\}$ is abelian
- (ii) $\{\tilde{\rho}_j^{II}\}$ is abelian

then $\rho^{\frac{1}{2}}$ is separable.

Proof. Suppose that there exists decomposition $\sum_j \tilde{\lambda}_j \tilde{\rho}_j^I \otimes \tilde{\rho}_j^{II}$ such that $\{\tilde{\rho}_j^I\}$ is abelian (the proof for the case when (ii) holds is similar). Let $\{\varphi_i\}$ be the orthonormal basis of \mathcal{H}_1 , in which $\tilde{\rho}_j^I$ are diagonal. Then, there exists decomposition $\rho = \sum_k \hat{\lambda}_k \hat{\rho}_k^I \otimes \hat{\rho}_k^{II}$ such that $\hat{\rho}_k^I = |\varphi_k\rangle\langle\varphi_k|$. Define matrix $\bar{\rho}$ as follows:

$$\bar{\rho} := \sum_k \hat{\lambda}_k^{\frac{1}{2}} \hat{\rho}_k^I \otimes (\hat{\rho}_k^{II})^{\frac{1}{2}}$$

Note that $\bar{\rho}$ is a linear combination with positive coefficients and matrices $\hat{\rho}_k^I$ and $\hat{\rho}_k^{II}$ are positive operators. To complete the proof we must show that $\bar{\rho}$ is the square root of ρ . We have:

$$\begin{aligned} \bar{\rho} \cdot \bar{\rho} &= \left(\sum_k \hat{\lambda}_k^{\frac{1}{2}} \hat{\rho}_k^I \otimes (\hat{\rho}_k^{II})^{\frac{1}{2}} \right) \cdot \left(\sum_l \hat{\lambda}_l^{\frac{1}{2}} \hat{\rho}_l^I \otimes (\hat{\rho}_l^{II})^{\frac{1}{2}} \right) = \sum_{kl} \hat{\lambda}_k^{\frac{1}{2}} \hat{\lambda}_l^{\frac{1}{2}} \cdot \hat{\rho}_k^I \hat{\rho}_l^I \otimes (\hat{\rho}_k^{II})^{\frac{1}{2}} (\hat{\rho}_l^{II})^{\frac{1}{2}} \\ &= \sum_{kl} \hat{\lambda}_k^{\frac{1}{2}} \hat{\lambda}_l^{\frac{1}{2}} \cdot \hat{\rho}_k^I \delta_{kl} \otimes (\hat{\rho}_k^{II})^{\frac{1}{2}} (\hat{\rho}_l^{II})^{\frac{1}{2}} = \sum_k \hat{\lambda}_k \hat{\rho}_k^I \otimes \hat{\rho}_k^{II} = \rho \end{aligned}$$

Note that the above sufficient conditions for the separability of the square root of ρ are essentially weaker than those for factorization of the correlation function.

5 Conclusions

Our analysis yields new information about the nature of quantum spin-flip dynamics. It was shown that this type of evolution can lead to entangled states, so to the family of states encoding quantum correlations. Furthermore, the detailed analysis of factorization property of the Ursell functions clearly shows that this is an expected phenomenon. It would be desirable to have the full description of evolution of entanglement but we have not been able to do this. The main difficulty in carrying out such a description is that we do not know the general characterization of interactions causing the spin-flip operation. However, we were able to give a detailed analysis of factorization of two-point correlation functions. We should also emphasize that in our case there is no point in considering such notions as decoherence. Although the region Λ_I over which we perform a local operation (e.g. a block-spin flip) can be macroscopic, we do not deal with the collective observables. This means that studying the evolution of entanglement as well as the correlation functions is the proper tool in investigating genuine microscopic properties of the considered quantum dynamics. These two approaches, studied in Section III and IV, respectively, are not equivalent. It is well known that if at least one of the two subsystems is a classical system (i.e. the underlying algebra of operators is commutative), there is no entanglement, even if the second subsystem is a purely quantum one [15]. On the other hand, studying the correlation functions allows us to "detect" quantum properties of the block-spin flip dynamics also when either of the two subsystems is a classical one (cf. Proposition 4.3). Thus, in that aspect, the correlations based approach is a more "subtle" tool as far as we examine the problem of the considered dynamics being the genuine quantum map or not. However, even the detailed analysis of the second Ursell function (the analysis of the third, fourth, ... Ursell functions is a very difficult task) can not supply enough information about the time evolution of states of the system. From this point of view the approach given in Section III is much more fruitful.

Turning back to the block-spin flip dynamics, we pointed out that its Markov generator should encode coupling between the region Λ_I and its environment Λ_{II} . Indeed, the results of Section III and IV say that the considered dynamics leads to enhancement of correlations. This means that the effect caused by the block-spin flip operation is strong, and it leads to coupling between two subsystems, therefore to nontrivial interactions. This enables us to interpret our result as another evidence that L_p -approach to quantum dynamics is working well in the sense that it leads to a fruitful recipe for explicit construction of interesting genuine quantum counterparts of classical dynamical maps.

Finally, we would like to remark that our results have been obtained for a low-dimensional model. Therefore, the expected and described properties of block-spin flip type dynamics follow exclusively from the noncommutativity of the underlying algebra of operators and have nothing to do with any transition from a finite to an infinite model via thermodynamic limit. Furthermore, we would like to emphasize that the presented theory has a fairly straightforward generalization to the infinite dimensional case as well as to other quantum jump processes.

Acknowledgement: It is a pleasure to thank M. Marciniak for inspiring discussions. The work of (W.A.M) has been supported by KBN grant PB/0273/PO/99/16.

Appendix A

For the convenience of the reader we recall the definition of separable and entangled states in the general setting of Hilbert spaces. The density matrix ρ (state) on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is called separable if it can be written or approximated (in the norm) by the density matrices (states) of the form:

$$\rho = \sum_i p_i \rho_i^1 \otimes \rho_i^2, \quad \left(\omega(\cdot) = \sum_i p_i (\omega_i^1 \otimes \omega_i^2)(\cdot) \right)$$

where $p_i \geq 0$, $\sum_i p_i = 1$, ρ_i^α are density matrices on \mathcal{H}_α , $\alpha = 1, 2$, and $(\omega_i^1 \otimes \omega_i^2)(A \otimes B) \equiv \omega_i^1(A) \cdot \omega_i^2(B) \equiv (Tr \rho_i^1 A) \cdot (Tr \rho_i^2 B) \equiv Tr\{\rho_i^1 \otimes \rho_i^2 \cdot A \otimes B\}$.

Definition 6.1 Let ρ be the separable state on $\mathcal{H}_1 \otimes \mathcal{H}_2$. We say that every finite sum of the form $\rho = \sum_i \lambda_i \rho_i^1 \otimes \rho_i^2$ is a decomposition of the state ρ iff

$$\begin{aligned} (i) \quad & \sum_i \lambda_i = 1 \quad \forall_i \lambda_i > 0 \\ (ii) \quad & \sum_i \lambda_i \rho_i^1 \otimes \rho_i^2 = \rho \end{aligned}$$

The state which is not separable is called non-separable. Denote by S the set of all states on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Definition 6.2 Non-separable states are called entangled states. The set of entangled states is defined by

$$S_{entangled} \equiv S \setminus S_{sep}$$

where S_{sep} stands for the set of separable states.

For a discussion of physical aspects of that concept see [18], [17], [2].

Appendix B

In this appendix we give the proof of Proposition 4.5. Let us introduce the following notation: $S_{nd} \subset \tilde{S}_{sep}$, $\rho \in S_{nd}$ if and only if the eigenvalues of $\tilde{\rho} = Tr_1(\rho)$ are not degenerated, $S_{nf} \subset \tilde{S}_{sep}$, $\rho \in S_{nf}$ if and only if $\langle E(f)g \rangle_\rho$ can not be factorized, $S_{ndf} := S_{nd} \cap S_{nf}$.

Lemma 7.1 *Let $\dim \mathcal{H}_1, \dim \mathcal{H}_2 < \infty$. Then the set S_{nd} is dense in \tilde{S}_{sep} in uniform topology (equivalently, it is dense in any operator topology).*

Proof. Let $\rho \in \tilde{S}_{sep}$ and $\rho = \sum_{i=1}^N \lambda_i \rho_i^I \otimes \rho_i^{II}$ be some decomposition of ρ . Let $\{\phi_j\}_{j=1}^m$ be the orthonormal basis of \mathcal{H}_2 such that $\tilde{\rho} = Tr_1(\rho)$ is diagonal. Denote by e_i the eigenvalue of $\tilde{\rho}$ corresponding to eigenvector ϕ_i . Of course, we have $e_i := \langle \phi_i | \tilde{\rho} | \phi_i \rangle$. Without loss of generality we can assume that only one eigenvalue is degenerated. In particular, we can assume that $e_1 = e_2$. Let $\epsilon > 0$. We will show that there exists $\hat{\rho} \in S_{nd}$ such that $\|\rho - \hat{\rho}\| < \epsilon$. Take η such that $0 < \eta < \frac{1}{2} \min\{\epsilon, |e_3 - e_1|, \dots, |e_m - e_1|\}$. Define:

$$\forall_{i=1, \dots, N} \quad \hat{\lambda}_i := \lambda_i(1 - \eta), \quad \hat{\rho}_i^I := \rho_i^I, \quad \hat{\rho}_i^{II} := \rho_i^{II}$$

$$\hat{\lambda}_{N+1} := \eta, \quad \hat{\rho}_{N+1}^I := \frac{1}{\dim \mathcal{H}_1} id_{\mathcal{H}_1}, \quad \hat{\rho}_{N+1}^{II} := |\phi_1\rangle\langle\phi_1|$$

and

$$\hat{\rho} := \sum_{i=1}^{N+1} \hat{\lambda}_i \hat{\rho}_i^I \otimes \hat{\rho}_i^{II}$$

Note that $\hat{\rho}$ is a well defined density matrix on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Moreover, the reduced density matrix $Tr_1(\hat{\rho}) = \sum_{i=1}^{N+1} \hat{\lambda}_i \hat{\rho}_i^{II} = \tilde{\rho}(1 - \eta) + \eta |\phi_1\rangle\langle\phi_1|$ has only nondegenerated eigenvalues $\hat{e}_1 = e_1$, $\hat{e}_2 = e_2(1 - \eta)$, \dots , $\hat{e}_m = e_m(1 - \eta)$, so $\hat{\rho} \in S_{nd}$. The lack of degeneracy stems from the choice of η because for all $i = 3, \dots, m$ we have $|e_i - e_1| > 2\eta$ and, evidently, $|e_i \eta| \leq \eta$. Now, suppose that $\hat{e}_1 = \hat{e}_j$ for some $j \in \{3, \dots, m\}$. We have:

$$\begin{aligned} \hat{e}_1 = \hat{e}_j & \Leftrightarrow e_1 = e_j(1 - \eta) \Leftrightarrow e_1 - e_j = -e_j \eta \Rightarrow \\ & \Rightarrow |e_1 - e_j| = |e_j \eta| \Rightarrow 2\eta < \eta \Leftrightarrow \eta < 0 \end{aligned}$$

which yields a contradiction, since η was assumed to be positive.

To complete the proof we must check that the inequality $\|\rho - \hat{\rho}\| < \epsilon$ holds. Indeed:

$$\begin{aligned} \|\rho - \hat{\rho}\| &= \left\| \sum_{i=1}^N \lambda_i \rho_i^I \otimes \rho_i^{II} - \sum_{i=1}^{N+1} \hat{\lambda}_i \hat{\rho}_i^I \otimes \hat{\rho}_i^{II} \right\| \\ &= \left\| \sum_{i=1}^N \lambda_i \rho_i^I \otimes \rho_i^{II} - \sum_{i=1}^N (1 - \eta) \lambda_i \rho_i^I \otimes \rho_i^{II} - \eta \hat{\rho}_{N+1}^I \otimes \hat{\rho}_{N+1}^{II} \right\| \\ &= \left\| \eta \sum_{i=1}^N \lambda_i \rho_i^I \otimes \rho_i^{II} - \eta \hat{\rho}_{N+1}^I \otimes \hat{\rho}_{N+1}^{II} \right\| \\ &= \eta \|\rho - \hat{\rho}_{N+1}^I \otimes \hat{\rho}_{N+1}^{II}\| \leq \eta (\|\rho\| + \|\hat{\rho}_{N+1}^I \otimes \hat{\rho}_{N+1}^{II}\|) \leq 2\eta < \epsilon \end{aligned}$$

Lemma 7.2 *Let $\dim \mathcal{H}_1, \dim \mathcal{H}_2 < \infty$. Then the set S_{ndf} is dense in S_{nd} in uniform topology (equivalently, it is dense in any operator topology).*

Proof. Let $\rho \in S_{nd}$ and $\dim \mathcal{H}_2 = m$. Suppose that $\langle E(f)g \rangle_\rho$ can be factorized. Then, from the relation between factorization and nondegeneracy of density matrix (see section 4.3) there exists decomposition $\rho = \sum_{i=1}^N \lambda_i \rho_i^I \otimes \rho_i^{II}$ such that $\{\rho_i^{II}\}$ is abelian. We can assume that N equals $\dim \mathcal{H}_2$ and $\rho_i^{II} = |\phi_i\rangle\langle\phi_i|$, where $\{\phi_j\}$ is the orthonormal basis of \mathcal{H}_2 such that $\tilde{\rho} = Tr_1(\rho)$ is abelian. Denote by e_i the eigenvalue of $\tilde{\rho}$ corresponding to the eigenvector ϕ_i (we have $e_i = \langle \phi_i | \tilde{\rho} | \phi_i \rangle$). Without loss of generality we can assume that eigenvalues of $\tilde{\rho}$ are ordered decreasingly, i.e. $e_1 > e_2 > \dots$.

Let $\epsilon > 0$. We will show that there exists $\hat{\rho} \in S_{ndf}$ such that $\|\rho - \hat{\rho}\| < \epsilon$. Take η such that $0 < \eta < \epsilon/2$. Let $\{\varphi_i\}_{i=1}^n$ be arbitrary but fixed orthonormal basis of \mathcal{H}_1 . Define:

$$\forall_{i=1,\dots,N} \quad \hat{\lambda}_i := \lambda_i(1-\eta), \quad \hat{\rho}_i^I := \rho_i^I, \quad \hat{\rho}_i^{II} := \rho_i^{II}$$

$$\hat{\lambda}_{N+1} := \frac{1}{2}\eta, \quad \hat{\rho}_{N+1}^I := |\varphi_1\rangle\langle\varphi_1|, \quad \hat{\rho}_{N+1}^{II} := \frac{1}{2}|\varphi_1\rangle\langle\varphi_1| + \frac{1}{2}|\varphi_2\rangle\langle\varphi_2| + \frac{i}{4}|\varphi_1\rangle\langle\varphi_2| - \frac{i}{4}|\varphi_2\rangle\langle\varphi_1|$$

$$\hat{\lambda}_{N+2} := \frac{1}{2}\eta, \quad \hat{\rho}_{N+2}^I := |\varphi_2\rangle\langle\varphi_2|, \quad \hat{\rho}_{N+2}^{II} := \frac{1}{2}|\varphi_1\rangle\langle\varphi_1| + \frac{1}{2}|\varphi_2\rangle\langle\varphi_2| - \frac{i}{4}|\varphi_1\rangle\langle\varphi_2| + \frac{i}{4}|\varphi_2\rangle\langle\varphi_1|$$

and

$$\hat{\rho} := \sum_{i=1}^{N+2} \hat{\lambda}_i \hat{\rho}_i^I \otimes \hat{\rho}_i^{II}$$

Note that $\hat{\rho}$ is well defined density matrix on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Moreover, $Tr_1(\hat{\rho}) = \sum_{i=1}^{N+2} \hat{\lambda}_i \hat{\rho}_i^{II} = \tilde{\rho}(1-\eta) + \frac{1}{2}\eta(|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|)$ has only nondegenerated eigenvalues $\hat{e}_1 = e_1(1-\frac{1}{2}\eta)$, $\hat{e}_2 = e_2(1-\frac{1}{2}\eta)$, $\hat{e}_3 = e_3(1-\eta)$, \dots , $\hat{e}_m = e_m(1-\eta)$, so $\hat{\rho} \in S_{nd}$.

Now we aim at showing that $\hat{\rho} \in S_{ndf}$. It is enough to show that there is no decomposition $\hat{\rho} = \sum_{i=1}^{\check{N}} \check{\lambda}_i \check{\rho}_i^I \otimes \check{\rho}_i^{II}$ for which $\{\check{\rho}_i^{II}\}$ is abelian, since due to the relation between factorization and nondegeneracy of density matrix (cf. section 4.3) it is equivalent to the fact that $\langle E(f)g \rangle_{\hat{\rho}}$ does not factorize. Suppose that there exists such a decomposition with $\{\check{\rho}_i^{II}\}$ abelian. We can assume that \check{N} equals $\dim \mathcal{H}_2$ and $\check{\rho}_i^{II} = |\phi_i\rangle\langle\phi_i|$. Then, we have:

$$\begin{aligned} \sum_{i=1}^{\check{N}} \check{\lambda}_i \check{\rho}_i^I \otimes \check{\rho}_i^{II} &= \sum_{i=1}^{N+2} \hat{\lambda}_i \hat{\rho}_i^I \otimes \hat{\rho}_i^{II} \\ \Rightarrow \sum_{i=1}^m \check{\lambda}_i \check{\rho}_i^I \otimes |\phi_i\rangle\langle\phi_i| &= \sum_{i=1}^m \hat{\lambda}_i \hat{\rho}_i^I \otimes |\phi_i\rangle\langle\phi_i| + \hat{\lambda}_{N+1} \hat{\rho}_{N+1}^I \otimes \hat{\rho}_{N+1}^{II} + \hat{\lambda}_{N+2} \hat{\rho}_{N+2}^I \otimes \hat{\rho}_{N+2}^{II} \\ \Rightarrow \sum_{i=1}^m (\check{\lambda}_i \check{\rho}_i^I - \hat{\lambda}_i \hat{\rho}_i^I) \otimes |\phi_i\rangle\langle\phi_i| &= \hat{\lambda}_{N+1} \hat{\rho}_{N+1}^I \otimes \hat{\rho}_{N+1}^{II} + \hat{\lambda}_{N+2} \hat{\rho}_{N+2}^I \otimes \hat{\rho}_{N+2}^{II} \end{aligned}$$

Since $(\check{\lambda}_i \check{\rho}_i^I - \hat{\lambda}_i \hat{\rho}_i^I) \otimes |\phi_i\rangle\langle\phi_i|$ are linearly independent, we have $\check{\lambda}_i \check{\rho}_i^I = \hat{\lambda}_i \hat{\rho}_i^I$ for $i = 3, \dots, m$, which leads to the following equality:

$$\sum_{i=1}^2 (\check{\lambda}_i \check{\rho}_i^I - \hat{\lambda}_i \hat{\rho}_i^I) \otimes |\phi_i\rangle\langle\phi_i| = \hat{\lambda}_{N+1} \hat{\rho}_{N+1}^I \otimes \hat{\rho}_{N+1}^{II} + \hat{\lambda}_{N+2} \hat{\rho}_{N+2}^I \otimes \hat{\rho}_{N+2}^{II}$$

Denote the left-hand side and the right-hand side of the above equality by L i P , respectively, we have:

$$\langle \varphi_1 \otimes \phi_1 | L | \varphi_1 \otimes \phi_2 \rangle = 0 \quad \text{and} \quad \langle \varphi_1 \otimes \phi_1 | P | \varphi_1 \otimes \phi_2 \rangle = \frac{i}{8}\eta \neq 0$$

which yields a contradiction. Thus, $\hat{\rho} \in S_{ndf}$.

To complete the proof we must check that the inequality $\|\rho - \hat{\rho}\| < \epsilon$ holds. Indeed:

$$\begin{aligned} \|\rho - \hat{\rho}\| &= \left\| \sum_{i=1}^N \lambda_i \rho_i^I \otimes \rho_i^{II} - \sum_{i=1}^{N+2} \hat{\lambda}_i \hat{\rho}_i^I \otimes \hat{\rho}_i^{II} \right\| \\ &= \left\| \sum_{i=1}^N \lambda_i \rho_i^I \otimes \rho_i^{II} - \sum_{i=1}^N (1-\eta) \lambda_i \rho_i^I \otimes \rho_i^{II} - \frac{1}{2}\eta \hat{\rho}_{N+1}^I \otimes \hat{\rho}_{N+1}^{II} - \frac{1}{2}\eta \hat{\rho}_{N+2}^I \otimes \hat{\rho}_{N+2}^{II} \right\| \\ &= \left\| \eta \sum_{i=1}^N \lambda_i \rho_i^I \otimes \rho_i^{II} - \frac{1}{2}\eta \hat{\rho}_{N+1}^I \otimes \hat{\rho}_{N+1}^{II} - \frac{1}{2}\eta \hat{\rho}_{N+2}^I \otimes \hat{\rho}_{N+2}^{II} \right\| \\ &= \eta \left\| \rho - \frac{1}{2} \hat{\rho}_{N+1}^I \otimes \hat{\rho}_{N+1}^{II} - \frac{1}{2} \hat{\rho}_{N+2}^I \otimes \hat{\rho}_{N+2}^{II} \right\| \\ &\leq \eta (\|\rho\| + \frac{1}{2} \|\hat{\rho}_{N+1}^I \otimes \hat{\rho}_{N+1}^{II}\| + \frac{1}{2} \|\hat{\rho}_{N+2}^I \otimes \hat{\rho}_{N+2}^{II}\|) \leq 2\eta < \epsilon \end{aligned}$$

Proof (of Proposition 4.5). The following inclusions hold: $S_{ndf} \subset S_{nd} \subset \tilde{S}_{sep}$. According to Lemma 7.1 and Lemma 7.2, S_{nd} is dense in \tilde{S}_{sep} and S_{ndf} is dense in S_{nd} , respectively. It means that S_{ndf} is dense in \tilde{S}_{sep} . Moreover, we have: $S_{ndf} \subset S_{nf} \subset \tilde{S}_{sep}$ which implies that S_{nf} is dense in \tilde{S}_{sep} . The proof is completed.

References

- [1] S. Albeverio, R. Höegh-Krohn, Commun. Math. Phys. **56**, 173 (1977)
- [2] Ch. Bennett, D.P. DiVincenzo, J.A. Smolin, W.K. Wootters, Phys. Rev. A **54**, 3824 (1996)
- [3] O. Bratteli, D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Springer Verlag, Vol.I (1979), Vol.II (1981)
- [4] F. Cipriani *Dirichlet forms and Markovian semigroups on standard forms on von Neumann algebras*, PhD Thesis, SISA, Trieste, (1992)
- [5] E.B. Davies, M. Lindsay, Math. Zeit. **210**, 379 (1992)
- [6] A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. **44**, 777 (1935)
- [7] S. Goldstein, M. Lindsay, Math. Zeit. **219**, 591 (1995)
- [8] R.V. Kadison, J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Academic Press, Vol.II, (1986)
- [9] T.M. Ligget, *Interacting Particle Systems*, Springer Verlag, (1985)
- [10] A.W. Majewski, B. Zegarlinski, Math. Phys. Electronic J. **1**, Paper 2 (1995)
- [11] A.W. Majewski, B. Zegarlinski, Lett. Math. Phys. **36**, 337 (1996)
- [12] A.W. Majewski, B. Zegarlinski, Rev. Math. Phys. **8**, 689 (1996)
- [13] A.W. Majewski, B. Zegarlinski, Markov Proc. and Rel. Fields **2**, 87 (1996)
- [14] A.W. Majewski, R. Olkiewicz, B. Zegarlinski, J. Phys. A. **31** 2045 (1998)
- [15] A.W. Majewski, Open Sys. & Information Dyn. **6**, 79 (1999)
- [16] A.W. Majewski, "On entanglement of states and quantum correlations" LANL preprint, math-ph/0202030
- [17] R. Omnès, *Understanding Quantum Mechanics*, Princeton University Press, (1999)
- [18] A. Peres, *Quantum Theory: Concepts and Methods*, Dordrecht, Kluwer (1995)
- [19] E. Schrödinger, Naturwissenschaften, **23**, 807, 823, 844 (1935)
- [20] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Berlin, Springer, (1932)